

Stability analysis of equilibria of linear delay complementarity systems

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Abstract—We present stability criteria for equilibria of a class of linear complementarity systems, subjected to discrete and distributed delay. We present necessary and sufficient conditions for local exponential stability, inferred from the spectrum location of a corresponding system of delay differential algebraic equations. Subsequently, we obtain sufficient LMI based conditions for global asymptotic stability using Lyapunov-Krasovskii functionals.

Index Terms—Stability of nonlinear systems; Delay systems; LMIs

I. INTRODUCTION

IN this paper, we study the stability properties of dynamical systems of which the dynamics are subject to delays and complementarity relationships.

On the one hand, delay systems have been widely studied and examples naturally arise in engineering, biology and control theory [7], [14], [15]. On the other hand, complementarity systems, being dynamical systems subject to complementarity conditions, have been used a.o. to model non-smooth mechanical systems [2], classes of hybrid systems [9] and power electronics converters [16]. However, systems with dynamics affected by both delays and complementarity conditions have been far less studied; these systems will be called *delay complementarity systems* in this paper.

Nevertheless, such systems also arise in a variety of applications, thereby motivating the further study of the stability properties of these systems. Firstly, delay complementarity models for directional drilling systems have recently been proposed in [10], [12]. Directional drilling models [5] describe the directional tendency of deep drilling systems used to generate complex curved boreholes in order to reach hard-to-access resources in the earth's crust. Delays appear in these models due to the fact that the deformed drill-string (inducing a directional tendency of the drilling bit) has to fit into a borehole generated in the recent past. Moreover, complementarity relations appear due to the non-smooth mechanics of the unilateral contact between the drill-string and bit on the one hand and the borehole on the other. The stability analysis

pursued in the current paper is essential to understand an instability phenomenon called borehole spiraling in such systems and to support controller design avoiding such instabilities. Another application in which delays and complementarity may appear is that of robotic teleoperation [11] in which delays are induced by networked communication between the robots and the unilateral contact between the robots and their environment can (in case of hard contacts) be described by complementarity relations.

The main contribution of this paper is the development of tools for (both the local and global) stability analysis of equilibria for a class of linear delay complementarity systems defined in more detail in the next section.

The structure of the paper is as follows. After introducing some notations, we introduce the considered class of delay complementarity systems and state the main assumptions in Section II, which induce the presence of a unique equilibrium. In Section III, we analyze local stability of this equilibrium, and in Section IV we propose conditions for the global stability. Finally, we illustrate the obtained stability criteria in Section V by means of an example.

Notations. Given $a, b \in \mathbb{R}^n$, let $a \perp b$ denote $a^T b = 0$ and a_i , for all $i \in \{1, 2, \dots, n\}$, denote the i -th element of the state a and $a > 0$ denotes $a_i > 0$ for $i = 1, 2, \dots, n$. For $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, \dots, x_s \in \mathbb{R}^{n_s}$, we denote the concatenated vector $(x_1^T \ x_2^T \ \dots \ x_s^T)^T$ as (x_1, x_2, \dots, x_s) . Given $m, n \in \mathbb{N}_{>0}$, let O_{mn} denote the zero matrix with dimension $m \times n$, I_n the n -dimensional identity matrix, and let the e_k , with $k \in \mathbb{N}_{>0}$, denote the k -th column of the identity matrix. For $M \in \mathbb{R}^{m \times m}$ and $x \in \mathbb{R}$, we use $\text{He}(M)$ to denote $M + M^T$, $\|x\|_M^2$ for $x^T M x$ and write $M \succ 0$ and $M \succeq 0$ if $\|x\|_M^2$ is positive or non-negative for all $x \neq 0$, respectively.

II. THE CLASS OF DELAY COMPLEMENTARITY SYSTEMS

We are interested in the stability properties of delay complementarity systems whose dynamics are described by

$$\begin{aligned} \dot{x}(t) = & A_0 x(t) + A_1 x(t - \tau) + A_2 \int_{t-\tau}^t x(s) ds \\ & + B_2 d + B_3 \lambda(t), \end{aligned} \quad (1a)$$

$$\begin{aligned} y(t) = & C_0 x(t) + C_1 x(t - \tau) + C_2 \int_{t-\tau}^t x(s) ds \\ & + D_2 d + D_3 \lambda(t), \end{aligned} \quad (1b)$$

$$0 \leq \lambda(t) \perp y(t) \geq 0, \quad (1c)$$

with $x \in \mathbb{R}^n$ the (instantaneous) state of the system, $d \in \mathbb{R}^s$ a constant disturbance, $\lambda \in \mathbb{R}^\ell$ forcing terms, associated with

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the complementarity constraints, and the variable \dot{x} denoting the right-sided derivative with respect to t . The dynamics of (1) are nonlinear due to the complementarity constraints (1c). These constraints mean that for each $i \in \{1, \dots, \ell\}$ and for each time-instant t , we have $\lambda_i(t) \geq 0$, $y_i(t) \geq 0$ and $y_i(t)\lambda_i(t) = 0$. It should be pointed out that as long as for each $i \in \{1, \dots, \ell\}$, we either have $y_i(t) > 0$ (a strictly open constraint) or $\lambda_i(t) > 0$ (a strictly closed constraint), Equations (1) are equivalent to (1a), supplemented with for each $i \in \{1, \dots, \ell\}$ either $\lambda_i(t) = 0$ or $y_i(t) = 0$, with y_i defined by (1b). The latter correspond to a system described by delay differential algebraic equations (DDAEs), also called a descriptor system with delay. Note, however, that along the trajectories constraints may open or close.

The initial condition of system (1) is given by an absolutely continuous function Φ on $[-\tau, 0]$ such that $x(s) = \Phi(s)$ for $s \in [-\tau, 0]$. Given a solution $x(t)$ to (1), for every $t \in \mathbb{R}_{\geq 0}$, we introduce the function $x_t : [-\tau, 0] \rightarrow \mathbb{R}^n$ such that $x_t(s) = x(t+s)$ for every $s \in [-\tau, 0]$.

In order to have a unique solution $\lambda(t)$ to (1c) for all initial functions $x_t(0)$, we adopt the following assumption:

Assumption 1. *Matrix D_3 is a P-matrix, i.e., every principal minor is strictly positive (see [3]).*

Equations (1) can be compactly written as

$$\dot{x} = \mathcal{A}(x_t) + B_3\lambda, \quad (2a)$$

$$0 \leq \lambda(t) \perp \mathcal{E}(x_t) + D_3\lambda(t) \geq 0, \quad (2b)$$

where we introduce the functionals $\mathcal{A}(x_t) = A_0x_t(0) + A_1x_t(-\tau) + A_2 \int_{s=-\tau}^0 x_t(s)ds + B_2d$ and $\mathcal{E}(x_t) = C_0x_t(0) + C_1x_t(-\tau) + C_2 \int_{s=-\tau}^0 x_t(s)ds + D_2d$. The vectors B_2d and D_2d imply that the complementarity part of this equation is a generalised linear complementarity system (generalized LCS), cf. [3]. The isolated and distributed delay terms in this expression introduce additional complexity.

Assumption 1 implies that for all $\mathcal{E} \in \mathbb{R}^\ell$ and all d , there exists a unique λ that is a solution to $0 \leq \lambda \perp \mathcal{E} + D_3\lambda \geq 0$, cf. (2b), which we denote by $\lambda = \text{SOL}(\mathcal{E}, D_3)$, and that is a piecewise linear function in \mathcal{E} , cf. [4, Proposition 1.4.6]. Hence, (2) allows the equivalent description as functional differential equation:

$$\dot{x}_t(0) = \mathcal{A}(x_t) + B_3\text{SOL}(\mathcal{E}(x_t), D_3), \quad (3)$$

where we define

$$\dot{x}_t(s) := \lim_{h \downarrow 0} \frac{x_t(s+h) - x_t(s)}{h}, \quad s \in [-\tau, 0).$$

We can directly observe from (2) that the system has equilibrium position x_e characterised by:

$$0 = (A_0 + A_1 + \tau A_2)x_e + B_2d + B_3\lambda_e, \quad (4a)$$

$$0 \leq \lambda_e \perp \beta_e \geq 0, \quad (4b)$$

where, for notational convenience, we introduce $\beta_e = (C_0 + C_1 + \tau C_2)x_e + D_2d + D_3\lambda_e$.

In the following sections, we investigate the stability of this equilibrium point (for stability notions for functional differential equations we refer to [7], [8]). For this purpose,

we introduce $\tilde{x} = x - x_e$ and $\tilde{\lambda} = \lambda - \lambda_e$ and observe that (2) is equivalent to

$$\dot{\tilde{x}} = A_0\tilde{x}(t) + A_1\tilde{x}(t-\tau) + A_2 \int_{t-\tau}^t \tilde{x}(s)ds + B_3\tilde{\lambda}, \quad (5a)$$

$$0 \leq \tilde{\lambda} + \lambda_e \perp \beta_e + C_0\tilde{x}(t) + C_1\tilde{x}(t-\tau) + C_2 \int_{t-\tau}^t \tilde{x}(s)ds + D_3\tilde{\lambda} \geq 0. \quad (5b)$$

For each t , we define $\tilde{x}_t : [-\tau, 0] \rightarrow \mathbb{R}^n$ such that $\tilde{x}_t(s) = x_t(s) - x_e$ for $s \in [-\tau, 0]$ and observe that this implies $\tilde{x}(t+s) = \tilde{x}_t(s)$ for all $s \in [-\tau, 0]$. In addition, we define the constant function $\tilde{x}_e : [-\tau, 0] \rightarrow \mathbb{R}^n$ as $\tilde{x}_e(s) \equiv x_e$.

III. LOCAL STABILITY ANALYSIS

In many cases, for each $k \in \{1, \dots, \ell\}$ (recall that ℓ is the dimension of λ , i.e., the number of constraints), we have that either $e_k^T \beta_e$ or $e_k^T \lambda_e$ are non-zero, i.e., at the equilibrium position x_e , each complementarity constraint is either strictly open, or strictly closed (closed with a nonzero constraint force), respectively. (This condition holds generically when $d \neq 0$.) In such cases, the local stability properties of the equilibrium point can be analysed using a spectral approach.

To do so, we introduce the $\ell \times \ell$ -dimensional diagonal matrix E^c with $E_{kk}^c = 1$ if $e_k^T \beta_e = 0$ and $E_{kk}^c = 0$ otherwise. We then find that the local dynamics near the equilibrium point x_e of (2) is determined by a descriptor system, described by the DDAE governed by (5a) and

$$0 = \begin{pmatrix} E^c & O_{\ell\ell} \\ O_{\ell\ell} & I_\ell - E^c \end{pmatrix} \begin{pmatrix} C_0\tilde{x}(t) + C_1\tilde{x}(t-\tau) + C_2 \int_{t-\tau}^t \tilde{x}(s)ds + D_3\tilde{\lambda} \\ \tilde{\lambda} \end{pmatrix}, \quad (6)$$

where we exploited (4) and the definition of the matrix E^c to infer $E^c \beta_e = 0$ and $(I_\ell - E^c)\lambda_e = 0$.

After removing the zero rows in (6), the resulting DDAE has index one (a semi-explicit equation), inferred from Assumption 1, and it satisfies the assumptions made in [13]. Accordingly, the behaviour of solutions near the equilibrium point of (2) and (6) and the local stability properties can be characterised by the location of the spectrum, providing necessary and sufficient stability conditions. In particular the equilibrium is locally exponentially stable if and only if all characteristic roots are located in the open left half complex plane. The latter can be obtained by solving the nonlinear eigenvalue problem in (μ, v) ,

$$\begin{pmatrix} -\mu \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A_0 & B_3 \\ \hat{E}^c C_0 & \hat{E}^c D_3 \end{pmatrix} \\ \begin{pmatrix} A_1 & 0 \\ \hat{E}^c C_1 & 0 \end{pmatrix} e^{-\mu\tau} + \begin{pmatrix} A_2 & 0 \\ \hat{E}^c C_2 & 0 \end{pmatrix} \frac{1-e^{-\mu\tau}}{\mu} \end{pmatrix} v = 0, \quad (7)$$

where \hat{E}^c , respectively \hat{F}^c , are obtained by removing the zero rows in E^c , respectively $I_\ell - E^c$. A dedicated approach for computing all eigenvalues in a prescribed right half plane is outlined in Section 4 of article [13], which is accompanied by publicly available software.

As this DDAE-approach ignores constraints that are non-active at the equilibrium point, only local stability results can be attained. In the following section, sufficient stability

conditions are given in terms of Lyapunov functionals. While this approach yields only sufficient conditions for stability, global results can be attained. In addition, also equilibrium positions x_e can be studied for which not all constraints are either strictly open, or closed with a nonzero constraint force.

IV. LYAPUNOV-KRASOVSKII FUNCTIONAL ANALYSIS

Global asymptotic stability of the equilibrium x_e of the system (2), and, equivalently, of the origin for the system (5), can be verified by the existence of a Lyapunov-Krasovskii functional V as specified the following lemma, that can be found e.g. in [8, Theorem 3.2 and Corollary 3.1]. To formulate the lemma, for given continuous functional V and solution \tilde{x}_t to the delay complementarity system (5), we define $\dot{V}(\tilde{x}_t) = \lim_{h \searrow 0} \frac{1}{h}(V(\tilde{x}_{t+h}) - V(\tilde{x}_t))$, and use the notation $\|\cdot\|_s$ for the supremum norm, i.e., $\|\phi\|_s = \sup_{s \in [-\tau, 0]} \|\phi(s)\|_2$.

Lemma 1. *The origin of (5) is globally asymptotically stable if there exist a continuous functional V , nonnegative and radially unbounded functions a and c , and a positive definite function b such that a) $c(\|\tilde{x}_t\|_s) \geq V(\tilde{x}_t) \geq a(\|\tilde{x}_t(0)\|)$ for all x_t and b) the functional $V(\tilde{x}_t)$, when evaluated along solutions to (2), satisfies $\dot{V}(\tilde{x}_t) \leq -b(\|\tilde{x}_t(0)\|)$.*

A. Design of the Lyapunov-Krasovskii functional

Following [3], we observe that the variable $\tilde{\lambda}$, when evaluated along a solution, becomes a function $\tilde{\lambda}(t)$ for which the right-derivative exists almost everywhere, and given (5), this right-sided derivative depends only on the instantaneous state \tilde{x}_t . We denote this right limit as $\tilde{\lambda}'$ and define the map $\text{SOLT}(\tilde{x}_t) : AC([-\tau, 0], \mathbb{R}^n) \mapsto \mathbb{R}^\ell \times \mathbb{R}^\ell$ such that

$$\text{SOLT}(\tilde{x}_t) \equiv \begin{pmatrix} \text{SOL}(\mathcal{E}(\tilde{x}_t + x_e), D_3) - \lambda_e \\ \tilde{\lambda}'(\tilde{x}_t) \end{pmatrix}. \quad (8)$$

With GrSOLT denoting the graph of this function, we will now provide conditions that guarantee that there exists a Lyapunov-Krasovskii functional V such that $\dot{V}(\tilde{x}_t) < 0$ for all $(\tilde{x}_t, \tilde{\lambda}, \tilde{\lambda}') \in \text{GrSOLT}$. We apply the \mathcal{S} -procedure to attain LMI conditions that guarantee these properties.

In [3], in the context of delay-free complementarity systems, a Lyapunov function is proposed that is a quadratic term of the concatenation of $x(t)$ and $\text{SOL}(\mathcal{E}(x(t)), D_3)$, i.e.

$$V_C(x(t)) = \begin{pmatrix} x(t) \\ \text{SOL}(\mathcal{E}(x(t)), D_3) \end{pmatrix}^T \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \begin{pmatrix} x(t) \\ \text{SOL}(\mathcal{E}(x(t)), D_3) \end{pmatrix},$$

such that a piecewise quadratic Lyapunov function in x is attained. Inspired by this approach and in pursuit of the extension to delay complementarity systems, we define

$$V_0(\tilde{x}_t) = \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda}(\tilde{x}_t) \end{pmatrix}^T \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda}(\tilde{x}_t) \end{pmatrix},$$

with $\tilde{\lambda}(x_t) = \text{SOL}(\mathcal{E}(\tilde{x}_t + x_e), D_3) - \lambda_e$. Similar to [1], we then design the continuous Lyapunov-Krasovskii functional as follows,

$$\begin{aligned} V(\tilde{x}_t) &= V_0(\tilde{x}_t) + \int_{-\tau}^0 \tilde{x}_t^T(s) S_1 \tilde{x}_t(s) ds \\ &+ \int_{-\tau}^0 \int_{\theta}^0 \tilde{x}_t^T(s) S_2 \tilde{x}_t(s) ds d\theta + \int_{-\tau}^0 \dot{\tilde{x}}_t(s)^T S_3 \dot{\tilde{x}}_t(s) ds \\ &+ \int_{-\tau}^0 \int_{\theta}^0 \dot{\tilde{x}}_t(s)^T S_4 \dot{\tilde{x}}_t(s) ds d\theta, \end{aligned} \quad (9)$$

where $P, S_1, S_2, S_3, S_4 \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{\ell \times \ell}$ are symmetric matrices, $Q \in \mathbb{R}^{n \times \ell}$ and $S_1, S_2, S_3, S_4 \succeq 0$. Further conditions on P, Q and R to ensure positive definiteness are discussed in Section IV-C. The used integral terms in \tilde{x}_t and $\dot{\tilde{x}}_t$ in (9) have originally been designed for linear delay system analysis without complementarity, see e.g. [6].

B. Evaluation of Lyapunov-Krasovskii functional along solutions

To pursue sufficient conditions for global exponential stability of the equilibrium point x_e for (2), we now present sufficient conditions for item b) of Lemma 1, i.e. for $\dot{V}(\tilde{x}_t) \leq -b(\|\tilde{x}_t(0)\|)$ with b a positive definite function, holding for almost all t when \tilde{x}_t satisfies (5). For notational convenience, given \tilde{x}_t , we introduce the vector $\zeta = (\tilde{x}(t), \dot{\tilde{x}}(t), \dot{\tilde{x}}(t - \tau), \tilde{\lambda}(t), \tilde{\lambda}'(t), 1)$.

Lemma 2. *Let $b_1 \in \mathbb{R}_{>0}$. The Lyapunov-Krasovskii functional $V(\tilde{x}_t)$ in (9) satisfies, along the solutions of the complementarity system (2),*

$$\begin{aligned} \dot{V}(\tilde{x}_t) + b_1 V_0(\tilde{x}_t) &= \zeta^T \Phi \zeta - \int_{t-\tau}^t \dot{\tilde{x}}^T(s) ds S_1 \int_{t-\tau}^t \dot{\tilde{x}}(s) ds \\ &+ 2\tilde{x}^T(s) S_1 \int_{t-\tau}^t \dot{\tilde{x}}(s) ds - \int_{t-\tau}^t \tilde{x}^T(s) S_2 \tilde{x}(s) ds \\ &- \int_{t-\tau}^t \dot{\tilde{x}}^T(s) S_4 \dot{\tilde{x}}(s) ds \end{aligned} \quad (10)$$

for all $(\tilde{x}_t, \tilde{\lambda}, \tilde{\lambda}') \in \text{GrSOLT}'$, where we introduced

$$\Phi = \begin{pmatrix} \tau S_2 + b_1 P & P + b_1 Q & O & O & Q & O \\ P^T + b_1 Q^T & S_3 + \tau S_4 + b_1 R & O & Q & O & O \\ O & O & -S_3 & O & O & O \\ O & Q^T & O & O & R & O \\ Q^T & O & O & O & O & O \\ O & O & O & O & O & O \end{pmatrix}. \quad (11)$$

Proof. Substituting the definition of the function \tilde{x}_t into (9), we find:

$$\begin{aligned} V(\tilde{x}_t) &= \begin{pmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{pmatrix}^T \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \begin{pmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{pmatrix} \\ &+ \int_{t-\tau}^t \tilde{x}^T(s) S_1 \tilde{x}(s) ds + \int_{-\tau}^0 \int_{t+\theta}^t \tilde{x}^T(s) S_2 \tilde{x}(s) ds d\theta \\ &+ \int_{t-\tau}^t \dot{\tilde{x}}(s)^T S_3 \dot{\tilde{x}}(s) ds + \int_{-\tau}^0 \int_{t+\theta}^t \dot{\tilde{x}}(s)^T S_4 \dot{\tilde{x}}(s) ds d\theta. \end{aligned} \quad (12)$$

Differentiating this expression and substituting $\tilde{x}(t - \tau) = \tilde{x}(t) - \int_{t-\tau}^t \dot{\tilde{x}}(s) ds$, we obtain:

$$\begin{aligned} \dot{V}(\tilde{x}_t) &= 2 \begin{pmatrix} \tilde{x}(t) \\ \tilde{\lambda}(t) \end{pmatrix}^T \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \begin{pmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{\lambda}}(t) \end{pmatrix} \\ &- \int_{t-\tau}^t \dot{\tilde{x}}^T(s) ds S_1 \int_{t-\tau}^t \dot{\tilde{x}}(s) ds + 2\tilde{x}^T(t) S_1 \int_{t-\tau}^t \dot{\tilde{x}}(s) ds \\ &+ \int_{-\tau}^0 (\tilde{x}^T(t) S_2 \tilde{x}(t) - \tilde{x}^T(t + \theta) S_2 \tilde{x}(t + \theta)) d\theta \\ &+ \dot{\tilde{x}}(t)^T S_3 \dot{\tilde{x}}(t) - \dot{\tilde{x}}(t - \tau)^T S_3 \dot{\tilde{x}}(t - \tau) \\ &+ \int_{-\tau}^0 (\dot{\tilde{x}}^T(t) S_4 \dot{\tilde{x}}(t) - \dot{\tilde{x}}^T(t - \theta) S_4 \dot{\tilde{x}}(t - \theta)) d\theta, \end{aligned}$$

which, by adding $b_1 V_0$ to left- and right-hand side, corresponds to (10). \square

In Lemma 2, we have not yet exploited any information of the set GrSOLT. To do so, we use the \mathcal{S} -procedure and add terms to the right-hand side of (10) which are nonnegative on GrSOLT. These terms are designed as follows.

- **Term 1:** The system dynamics (5a) are equivalent to the statement $0 = -\dot{\tilde{x}}(t) + (A_0 + A_1)\tilde{x}(t) + (A_2 - A_1) \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds + B_3 \tilde{\lambda}(t)$. Pre-multiplication with the vector $2\zeta^T T_0$ yields

$$0 = \zeta^T \Psi_0 \zeta + 2\zeta^T \eta_0 \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds, \quad (13)$$

with $\Psi_0 = \text{He} \begin{pmatrix} T_0 (A_0 + A_1 & -I & O & B_3 & O & O) \\ T_0 (A_2 & -A_1) \end{pmatrix}$ and $\eta_0 = T_0 \begin{pmatrix} A_2 & -A_1 \end{pmatrix}$. This expression holds for all matrices $T_0 = (T_{00}^T \ T_{01}^T \ \cdots \ T_{05}^T)^T \in \mathbb{R}^{(3n+2\ell+1) \times n}$.

- **Term 2:** The condition $\tilde{\lambda} + \lambda_e \perp \beta_e + C_0 \tilde{x}(t) + C_1 \tilde{x}(t - \tau) + C_2 \int_{t-\tau}^t \tilde{x}(s) ds + D_3 \tilde{\lambda}$ in (5b) is equivalent to the condition that $2(\tilde{\lambda} + \lambda_e)^T T_1 (\beta_e + C_0 \tilde{x}(t) + C_1 \tilde{x}(t - \tau) + C_2 \int_{t-\tau}^t \tilde{x}(s) ds + D_3 \tilde{\lambda}) = 0$ holds for any diagonal matrix $T_1 \in \mathbb{R}^{\ell \times \ell}$. Rewriting gives

$$\begin{aligned} 0 &= 2\tilde{\lambda}^T T_1 D_3 \tilde{\lambda} + 2\tilde{\lambda}^T \left(T_1 (\beta_e + (C_0 + C_1)\tilde{x}(t) \right. \\ &\quad \left. + (C_2 - C_1) \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds) + D_3^T T_1 \lambda_e \right) \\ &\quad + 2\lambda_e^T T_1 (\beta_e + (C_0 + C_1)(\tilde{x}(t)) + (C_2 - C_1) \\ &\quad \left. \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \right). \end{aligned}$$

Hence, we find

$$0 = \zeta^T \Psi_1 \zeta + 2\zeta^T \eta_1 \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds, \quad (14)$$

with

$$\Psi_1 = \text{He} \begin{pmatrix} \begin{array}{c|c} \begin{matrix} O_{3n,3n} \\ T_1(C_0 + C_1) & O & O \\ O & O & O \\ \lambda_e^T T_1(C_0 + C_1) & O & O \end{matrix} & \begin{matrix} O_{3n,2\ell+1} \\ T_1 D_3 & O \\ O & O & O \\ O & O & O \end{matrix} \end{array} \begin{matrix} T_1 \beta_e + \\ D_3^T T_1 \lambda_e \\ O \end{matrix} \end{pmatrix},$$

$$\eta_1 = \begin{pmatrix} \begin{array}{c|c} \begin{matrix} O_{3n,2n} \\ T_1 C_2 & -T_1 C_1 \\ O & O \\ \lambda_e^T T_1 C_2 & -\lambda_e^T T_1 C_1 \end{matrix} & \end{array} \end{pmatrix} \text{ and where we ex-}$$

ploited the observation that (4) implies $\lambda_e^T T_1 \beta_e = 0$.

- **Term 3:** The inequalities in (5b) imply $\Xi \zeta + \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix} \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \geq 0$, with $\Xi = \begin{pmatrix} C_0 + C_1 & O & O & D_3 & O & \beta_e \\ O & O & O & I & O & \lambda_e \end{pmatrix}$. As $u^T T_2 u \geq 0$ if $u \geq 0$ for any symmetric matrix $T_2 \in \mathbb{R}^{2\ell \times 2\ell}$ with nonnegative elements, we find:

$$\begin{aligned} 0 &\leq \zeta^T \Psi_2 \zeta + 2\zeta^T \eta_2 \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \\ &\quad + \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T ds \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix} \\ &\quad \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds, \end{aligned} \quad (15)$$

with $\Psi_2 = \Xi^T T_2 \Xi$ and $\eta_2 = \Xi^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix}$.

- **Term 4:** We note that the complementarity relation (5b) implies that, for each $k \in \{1, \dots, \ell\}$, either $\tilde{\lambda}'_k(t) = 0$

(recall the definition of λ' in the beginning of Section IV-A) or

$$\begin{aligned} &e_k^T (\beta_e + C_0 \tilde{x}(t) + C_1 \tilde{x}(t - \tau) \\ &+ C_2 \int_{t-\tau}^t \tilde{x}(s) ds + D_3 \tilde{\lambda}) = 0. \end{aligned}$$

Hence, we find

$$\begin{aligned} &\tilde{\lambda}'^T T_3 (\beta_e + C_0 \tilde{x}(t) + C_1 \tilde{x}(t - \tau) \\ &+ C_2 \int_{t-\tau}^t \tilde{x}(s) ds + D_3 \tilde{\lambda}) = 0 \end{aligned}$$

for any diagonal $T_3 \in \mathbb{R}^{\ell \times \ell}$ and

$$0 = \zeta^T \Psi_3 \zeta + 2\zeta^T \eta_3 \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \quad (16)$$

holds with

$$\Psi_3 = \text{He} \begin{pmatrix} E_5 T_3 (C_0 + C_1 & O & O & D_3 & O & \beta_e) \end{pmatrix}$$

and $\eta_3 = E_5^T T_3 (C_2 - C_1)$, where $E_5 = (O_{\ell,3n} \ O_{\ell} \ I_{\ell} \ O_{\ell,1})^T$.

- **Term 5:** From the complementarity relation (5b) it also follows that $0 = 2(\tilde{\lambda} + \lambda_e)^T T_4 (C_0 \tilde{x}(t) + C_1 \tilde{x}(t - \tau) + C_2 \int_{t-\tau}^t \tilde{x}(s) ds + D_3 \tilde{\lambda})$ holds for arbitrary diagonal T_4 . Hence, we find

$$0 = \zeta^T \Psi_4 \zeta + 2\zeta^T \eta_4 \int_{t-\tau}^t \begin{pmatrix} x(s) - x_e \\ \dot{x}(s) \end{pmatrix} ds, \quad (17)$$

with

$$\Psi_4 = \text{He} \begin{pmatrix} (O_{\ell,3n} \ I_{\ell} \ O_{\ell,\ell} \ \lambda_e)^T T_4 \\ (O \ C_0 \ C_1 \ O \ D_3 \ O) \end{pmatrix}$$

and $\eta_4 = (O_{\ell,3n} \ I_{\ell} \ O_{\ell,\ell} \ \lambda_e)^T T_4 (O \ C_2)$.

With the five \mathcal{S} -procedure terms given in the list above, we can provide matrix inequality conditions that guarantee the property in Lemma 2 for $(\tilde{x}_t, \tilde{\lambda}, \tilde{\lambda}') \in \text{GrSOLT}$ as follows.

Lemma 3. Consider matrices $T_0 \in \mathbb{R}^{(3n+2\ell+1) \times n}$, $T_1 \in \mathbb{R}^{\ell \times \ell}$, $T_2 \in \mathbb{R}^{(2\ell) \times (2\ell)}$, $T_3, T_4 \in \mathbb{R}^{\ell \times \ell}$ and $U \in \mathbb{R}^{2n \times 2n}$, with T_1, T_3, T_4 diagonal, T_2 symmetric with nonnegative elements, and $U \succ 0$. With c_{12} denoting an orthonormal matrix that spans the column space of $\begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix}$, we assume

$$c_{12}^T T_2 c_{12} \succ 0. \quad (18)$$

Evaluated along solutions to (2), the Lyapunov-Krasovskii functional (9) verifies $\dot{V}(\tilde{x}_t) \leq -b_1 V_0(\tilde{x}_t)$ if

$$\tau \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & O \end{pmatrix} + U - \begin{pmatrix} S_2 & O \\ O & S_4 \end{pmatrix} \prec 0 \quad (19)$$

and

$$\begin{pmatrix} \Psi & -\tau \eta \\ -\tau \eta^T & -\tau U \end{pmatrix} \preceq 0, \quad (20)$$

hold, where $\Psi = \Phi + \sum_{k=0}^4 \Psi_k$, with Φ depending on b_1 and defined in Lemma 2, $\eta = \mu + \sum_{k=0}^4 \eta_k$, with $\mu = 2 \begin{pmatrix} O_{nn} & S_1 \\ O_{2n+2\ell+1,n} & O_{2n+2\ell+1,n} \end{pmatrix}$.

Proof. We sum (10), (13), (14), (15), (16) and (17) to attain:

$$\begin{aligned} \dot{V} + b_1 V_0(\tilde{x}_t) &\leq \zeta^T \Psi \zeta + 2\zeta^T \eta \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \\ &+ \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T ds \left(\begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix} + \begin{pmatrix} O & -O_1 \\ O & -S_1 \end{pmatrix} \right) \\ &\int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds - \int_{t-\tau}^t \tilde{x}^T(s) S_2 \tilde{x}(s) ds - \int_{t-\tau}^t \dot{\tilde{x}}^T(s) S_4 \dot{\tilde{x}}(s) ds \end{aligned} \quad (21)$$

We will now upper bound the third term of the right-hand side of (21). With (18), we can apply Jensen's inequality (see, e.g. [7]) to find

$$\begin{aligned} &\int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T ds \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix} \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \\ &\leq \tau \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix} \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \end{aligned} \quad (22)$$

and as $S_1 \succeq 0$ by construction, removing the term $\begin{pmatrix} O & -O_1 \\ O & -S_1 \end{pmatrix}$ is allowed. To upper bound the second term in the right-hand side of (21), similar to [1], we use that for any $U = U^T \in \mathbb{R}^{n \times n}$, with $U \succ 0$:

$$\begin{aligned} &2\zeta^T \eta \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds \\ &\leq \int_{t-\tau}^t \left(\zeta^T \eta U^{-1} \eta^T \zeta + \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T U \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} \right) ds \quad (23) \\ &\leq \tau \zeta^T \eta U^{-1} \eta^T \zeta + \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T U \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds. \end{aligned}$$

Combining these overapproximations with (21), we find:

$$\begin{aligned} \dot{V} &\leq -b_1 V_0(\tilde{x}_t) + \zeta^T (\Psi + \tau \eta U^{-1} \eta^T) \zeta \\ &+ \int_{t-\tau}^t \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix}^T \left(\tau \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix}^T T_2 \begin{pmatrix} C_2 & -C_1 \\ O & -O_1 \end{pmatrix} \right. \\ &\left. + U - \begin{pmatrix} S_2 & O \\ O & S_4 \end{pmatrix} \right) \begin{pmatrix} \tilde{x}(s) \\ \dot{\tilde{x}}(s) \end{pmatrix} ds. \end{aligned} \quad (24)$$

Consequently, $\dot{V} \leq -b_1 V_0(\tilde{x}_t)$ holds if (19) holds as well as $\Psi + \tau \eta U^{-1} \eta^T \prec 0$ that, by the Schur complement, is equivalent to (20), concluding the proof. \square

C. Main result

Below, we formulate the main theorem of this paper that provides sufficient conditions for global asymptotic stability of the equilibrium position x_e of the delay complementarity system (2) in terms of LMIs. In this theorem, to ensure positive definiteness of the Lyapunov-Krasovskii functional as required in item a) of Lemma 1, two options are provided. The second option may provide less conservative results as the complementarity relation in (2b) is used, but is only applicable when $C_1 = 0, C_2 = 0$.

Theorem 4. Consider the delay complementarity system (2) and suppose there exist symmetric matrices P, Q and matrix R such that either

$$\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \succ 0$$

holds, or that $C_1 = 0, C_2 = 0$ and there exist matrices $T_5 \in \mathbb{R}^{\ell \times \ell}$, $T_6 \in \mathbb{R}^{2\ell \times 2\ell}$, with T_5 diagonal and T_6 symmetric and having non-negative elements, such that

$$\begin{aligned} &\begin{pmatrix} P & Q & O_{n1} \\ Q^T & R & O_{\ell 1} \\ O_{1\ell} & O_{1n} & 0 \end{pmatrix} - \text{He} \left(\begin{pmatrix} O_{\ell\ell} \\ I_\ell \\ \lambda_e^T \end{pmatrix} T_5 \begin{pmatrix} C_0 & D_3 & \beta_e \end{pmatrix} \right) \\ &- \begin{pmatrix} C_0 & D_3 & \beta_e \\ O_{\ell\ell} & I_\ell & \lambda_e \end{pmatrix}^T T_6 \begin{pmatrix} C_0 & D_3 & \beta_e \\ O_{\ell\ell} & I_\ell & \lambda_e \end{pmatrix} \succeq \begin{pmatrix} I_{n+\ell} & O_{n+\ell,1} \\ O_{n+\ell,1} & 0 \end{pmatrix}. \end{aligned} \quad (25)$$

If, in addition, there exist matrices $T_0, T_1, T_2, T_3, T_4, U$ as in Lemma 3 and matrices $S_1, S_2, S_3, S_4 \succeq 0$, which verify the LMIs (19) and (20) for some scalar $b_1 > 0$, then the equilibrium position x_e of (2) is globally asymptotically stable, and functional (9) satisfies $\dot{V}(\tilde{x}_t) \leq -b_1 V_0(\tilde{x}_t)$.

Proof. The proof of this theorem exploits Lemma 1 and we first construct the function $a(r)$ as in item a) of this lemma. If $\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \succ 0$, we select $a(r)$ as σr^2 , with σ the minimum eigenvalue of the matrix $\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}$, and observe $\sigma > 0$. With $S_1, S_2, S_3, S_4 \succeq 0$, we find $V(\tilde{x}_t) \geq V_0(\tilde{x}_t) = \left\| \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda}(\tilde{x}_t) \end{pmatrix} \right\|^2_{\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}} \geq \sigma \left\| \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda}(\tilde{x}_t) \end{pmatrix} \right\|^2 \geq \sigma \|\tilde{x}_t(0)\|^2 = a(\|\tilde{x}_t(0)\|)$.

If $C_1, C_2 = 0$ and (25) holds, then (5) yields $0 \leq \tilde{\lambda} + \lambda_e \perp \beta_e + C_0 \tilde{x}(t) + D_3 \tilde{\lambda} \geq 0$, which implies $2 \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda} \\ 1 \end{pmatrix}^T \begin{pmatrix} O_{\ell\ell} \\ I_\ell \\ \lambda_e^T \end{pmatrix} T_5 \begin{pmatrix} C_0 & D_3 & \beta_e \end{pmatrix} \begin{pmatrix} \tilde{x}_t(0) \\ \tilde{\lambda} \\ 1 \end{pmatrix} = 0$ since T_5 is diagonal. In addition, the mentioned complementarity inequality implies $\begin{pmatrix} x_t(0) - x_e \\ \tilde{\lambda} - \lambda_e \\ 1 \end{pmatrix}^T \begin{pmatrix} C_0 & D_3 & \beta_e \\ O_{\ell\ell} & I_\ell & \lambda_e \end{pmatrix}^T T_6 \begin{pmatrix} C_0 & D_3 & \beta_e \\ O_{\ell\ell} & I_\ell & \lambda_e \end{pmatrix} \begin{pmatrix} x_t(0) - x_e \\ \tilde{\lambda} - \lambda_e \\ 1 \end{pmatrix} \succeq 0$,

since T_6 has non-negative elements. Consequently, pre- and postmultiplication of (25) with $(\tilde{x}(t)^T \tilde{\lambda}(\tilde{x}_t)^T \ 1)$ and its transpose, respectively, directly implies $V_0(\tilde{x}_t) \geq \left\| \begin{pmatrix} \tilde{x}(t) \\ \tilde{\lambda}(\tilde{x}_t) \end{pmatrix} \right\|^2 \geq \|\tilde{x}(t)\|^2$. Since $V(\tilde{x}_t) \geq V_0(\tilde{x}_t)$, we can select $a(r) = r^2$ such that $V(\tilde{x}_t) \geq a(\|\tilde{x}_t(0)\|)$ is attained. In both cases of the theorem, we have designed a such that $V(\tilde{x}_t) \geq a(\|\tilde{x}_t(0)\|)$ follows. Hence, item a) of Lemma 1 is verified.

To verify item b) of Lemma 1, we observe that Lemma 3 yields $\dot{V}(\tilde{x}_t) \leq -b_1 V_0(\tilde{x}_t)$, which, with the result above, implies $\dot{V}(\tilde{x}_t) \leq -b(\|\tilde{x}_t(0)\|)$, with $b(r) = b_1 a(r)$, that is a positive definite function. Hence, all conditions in Lemma 1 are verified and global asymptotic stability of the origin of (5) follows. \square

V. EXAMPLE

We consider (1) with dimension $n = 2$, two constraints $\ell = 2$, delay $\tau = 0.1$ and system matrices given by

$$\begin{aligned} A_0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -1 & -0.4 \end{pmatrix}, \quad A_2 = 0_{2,2}, \\ B_2 &= \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 \\ 6 & -6 \end{bmatrix}, \quad C_0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

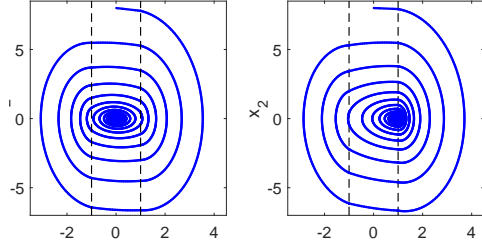


Fig. 1. A solution for $\alpha = 0$ (left pane) and $\alpha = 1$ (right pane).

$$C_1 = 0_{2,2}; \quad C_2 = 0_{2,2}, \quad D_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad D_3 = I_2,$$

where $\alpha \geq 0$ is a parameter. For this problem the complementarity constraints read as

$$0 \leq \lambda_1 \perp x_1 + 1 + \lambda_1 \geq 0, \quad 0 \leq \lambda_2 \perp -x_1 + 1 + \lambda_2 \geq 0,$$

which allow to express λ as a function of x as

$$\lambda_1 = \begin{cases} -x_1 - 1, & x_1 \leq -1, \\ 0, & \text{otherwise} \end{cases}, \quad \lambda_2 = \begin{cases} x_1 - 1, & x_1 \geq 1, \\ 0, & \text{otherwise} \end{cases}.$$

A computation based on (4) yields:

$$x_e = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \lambda_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \beta_e = \begin{pmatrix} 1 + \alpha \\ 1 - \alpha \end{pmatrix}, \quad \alpha \leq 1,$$

$$x_e = \begin{pmatrix} \frac{\alpha+6}{7} \\ 0 \end{pmatrix}, \quad \lambda_e = \begin{pmatrix} 0 \\ \frac{\alpha-1}{7} \end{pmatrix}, \quad \beta_e = \begin{pmatrix} \frac{\alpha+13}{7} \\ 0 \end{pmatrix}, \quad \alpha > 1.$$

A linearized stability analysis of the equilibrium, based on solving (7), yields that it is locally asymptotically stable, with spectral abscissa (real part of rightmost characteristic root) equal to -0.155 for $\alpha < 1$ and -0.149 for $\alpha > 1$. Note that for $\alpha = 1$ the second constraint is closed and the constraint force vanishes, so that the spectral approach cannot be used.

Global asymptotic stability is guaranteed if the LMI feasibility problem (19), (20) and (25) has a solution. This is for instance the case for $\alpha = 0$, induced by Lyapunov function (9), where

$$P = \begin{pmatrix} 2.21 & 0.415 \\ 0.415 & 2.13 \end{pmatrix}, \quad Q = 0_2, \quad R = \begin{pmatrix} 12.8 & 0 \\ 0 & 12.8 \end{pmatrix}, \\ S_1 = \begin{pmatrix} 0.531 & -0.0753 \\ -0.0753 & 0.0907 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 2.12 & -0.0185 \\ -0.0185 & 1.406 \end{pmatrix}, \\ S_3 = \begin{pmatrix} 0.117 & 0.00122 \\ 0.00122 & 0.0133 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 2.2345 & 0.3532 \\ 0.3532 & 0.4504 \end{pmatrix}.$$

In Figure 1 we depict the solution x with initial condition $\phi(s) = (0 \ 8)^T$, $s \in [-0.1, 0]$, for $\alpha = 0$ (left) and $\alpha = 1$ (right). In Figure 2, we show x_1 , λ_1 , λ_2 corresponding to the solution for $\alpha = 1$, illustrating that the first, respectively the second constraint becomes active if x_1 leave the interval $[-1, 1]$ from below, respectively from above.

VI. CONCLUSIONS

We have introduced a class of delay complementarity systems. Both necessary and sufficient conditions for the local asymptotic stability of equilibria are presented, and sufficient conditions for the global asymptotic stability.

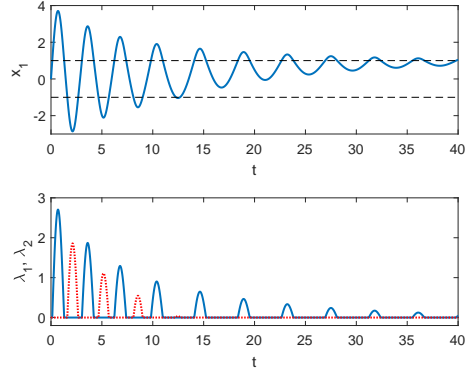


Fig. 2. Variables x_1 (upper pane), λ_1 (lower pane, red dashed curve) and λ_2 (lower pane, solid curve), corresponding to the solution for $\alpha = 1$ shown in Figure 1 (right).

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REFERENCES

- [1] J. J. B. Biemond and W. Michiels. Estimation of basins of attraction for controlled systems with input saturation and time-delays. *Systems & Control Letters*, pages 84–91, 2016.
- [2] B. Brogliato. *Nonsmooth Mechanics*. Springer-Verlag, London, 1999.
- [3] M. K. Camlibel, J.-S. Pang, and J. Shen. Lyapunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, 17(4):1056–1101, 2006.
- [4] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The linear complementarity problem*, volume 60 of *Classics in Applied Mathematics*. SIAM, Philadelphia, 2009.
- [5] E. Detournay, J. Marck, and L. Perneder. Mechanics of directional drilling. In *Rock Mechanics and Engineering*, chapter 4. CRC Press, 2016.
- [6] E. Fridman and U. Shaked. A descriptor system approach to H_∞ control of linear time-delay systems. *IEEE Transactions on Automatic Control*, 47(2):253–270, 2002.
- [7] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Control Engineering. Springer, 2003.
- [8] J. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.
- [9] W.P.M.H. Heemels and B. Brogliato. The complementarity class of hybrid dynamical systems. *European Journal of Control*, 26(4):651–677, 2003.
- [10] B. Heijke. A hybrid model of a two-dimensional directional drilling system. Master's thesis, Delft University of Technology, The Netherlands, 2016.
- [11] P.F. Hokayem and M.W. Spong. Bilateral teleoperation: an historical survey. *Automatica*, 42:2035–2057, 2006.
- [12] J. Marck. *A Nonlinear Dynamical Model of Borehole Spiraling*. PhD thesis, University of Minnesota, U.S.A., 2015.
- [13] W. Michiels. Spectrum based stability analysis and stabilization of systems described by delay differential algebraic equations. *IET Control Theory and Applications*, 5(16):1829–1842, 2011.
- [14] W. Michiels and S.-I. Niculescu. *Stability, Control and Computation for Time-Delay Systems: an eigenvalue approach*. SIAM, Philadelphia, 2nd edition, 2014.
- [15] G. Stepan. *Retarded Dynamical Systems: Stability and Characteristic Functions*. John Wiley and Sons, New York, U.S.A., 1989.
- [16] F. Vasca, L. Iannelli, M.K. Camlibel, and R. Frasca. A new perspective for modeling power electronics converters: Complementarity framework. *IEEE Transactions on Power Electronics*, 24(1-2):456–468, 2009.